The Hatcher Spectral Sequence

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What we do here is describe a spectral sequence and do a little clarification on what the differential is when we float between some isomorphisms. Let M be compact with a possibly non-empty boundary, note that [Hat78] deals explicitly only with the closed case. Let $\tilde{A}(M \times I^k) := \mathrm{Diff}_\partial(M \times I^k; \mathrm{proj}_{I_k} \text{ over } \partial I^k)/\mathrm{Diff}_\partial(M \times I^k; \mathrm{proj}_{I_k})$, that is diffeomorphisms of $M \times I^k$ *rel* $\partial M \times I^k$ that preserve the projection to I^k over ∂I^k. Let $\tilde{C}(M \times I^k) := \mathrm{Diff}_\partial(M \times I^k \times I; \mathrm{proj}_{I_k \times I} \text{ over } I^k \times 0 \cup \partial I^k \times I)/\mathrm{Diff}_\partial(M \times I^k \times I; \mathrm{proj}_{I_k \times I})$. These, basically simplicial groups, are ugly but up to homotopy familiar

Lemma ([Kup], Lem 25.3.6)**.** *$\tilde{C}(M \times I^k) \simeq \mathscr{C}(M \times I^k)$ the group of concordances.*

Hatcher says that "there are fibrations"

$$\tilde{A}(M \times I^{k+1}) \to \tilde{C}(M \times I^k) \to \tilde{A}(M \times I^k)$$

"which gives an exact couple"

$$\begin{tikzcd}
\Sigma_{j,k}\pi_j\tilde{A}(M \times I^k) \arrow[rr, "\partial"] & & \Sigma_{j,k}\pi_j\tilde{A}(M \times I^{k+1}) \arrow[dl] \\
& \sum_{j,k}\pi_j\tilde{C}(M \times I^k) \arrow[ul] &
\end{tikzcd}$$

To elucidate this, the exact couple is just rolling up the exact sequence and adding them all up. This makes the E^0 page simply these exact sequences layed vertically as below

$$\begin{tikzcd}
& \pi_1\mathscr{C}(M \times I^0) \arrow[d] & \pi_1\mathscr{C}(M \times I^1) \arrow[d] & \pi_1\mathscr{C}(M \times I^2) \arrow[d] & \\
& \pi_1\tilde{A}(M \times I^0) \arrow[d, "\partial"] & \pi_1\tilde{A}(M \times I^1) \arrow[d, "\partial"] & \pi_1\tilde{A}(M \times I^2) \arrow[d, "\partial"] & \\
& \pi_0\tilde{A}(M \times I^1) \arrow[d] & \pi_0\tilde{A}(M \times I^2) \arrow[d] & \pi_0\tilde{A}(M \times I^3) \arrow[d] & \\
& \pi_0\mathscr{C}(M \times I^0) \arrow[d] & \pi_0\mathscr{C}(M \times I^1) \arrow[d] & \pi_0\mathscr{C}(M \times I^2) \arrow[d] & \\
0 & \pi_0\tilde{A}(M \times I^0) \arrow[d] & \pi_0\tilde{A}(M \times I^1) \arrow[d] & \pi_0\tilde{A}(M \times I^2) \arrow[d] & \cdots \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Note that the red arrows in the LES of a fibration are not exact, that is the maps out of the final π_0 are not exact. The green groups form the actual E^0 page and because the maps are exact, and to go between two of them we go through two maps in the LES all the maps are zero. This is not true at the final peice, however we have alternative means to see that these maps must be zero, because for example we have Cerf telling us that $\pi_0\mathscr{C}(D^n) \cong 0$ for $n \geq 5$. I dont beleive that pi 0 of the A groups in general is zero, and neither is concordance so, when both are non-zero what can we say about this in that case. So the E^0 page is the following

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 & \pi_1\mathscr{C}(M \times I^0) \arrow[d, "0"] & \pi_1\mathscr{C}(M \times I^1) \arrow[d, "0"] & \pi_1\mathscr{C}(M \times I^2) \arrow[d, "0"] & \cdots \\
0 & \pi_0\mathscr{C}(M \times I^0) \arrow[d] & \pi_0\mathscr{C}(M \times I^1) \arrow[d] & \pi_0\mathscr{C}(M \times I^2) \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The next page is then the non-trivial page that Hatcher will give a nice geometric description of

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 & \pi_1\mathscr{C}(M \times I^0) \arrow[l] & \pi_1\mathscr{C}(M \times I^1) \arrow[l] & \pi_1\mathscr{C}(M \times I^2) \arrow[l] & \cdots \\
0 & \pi_0\mathscr{C}(M \times I^0) \arrow[l] & \pi_0\mathscr{C}(M \times I^1) \arrow[l] & \pi_0\mathscr{C}(M \times I^2) \arrow[l] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Now we will just be honest and consider $M = D^n$. Hatcher tells us that the differential is given by restriction to the upper boundary. Now **if we are in the stable range**, that is close enough to the bottom row for n large, then the groups on the horrozontals are all equal in a given row. So we can write the E^1 page as

$$\begin{tikzcd}
0 & \pi_2\mathscr{C}(D^n) \arrow[l] & \pi_2\mathscr{C}(D^n) \arrow[l] & \pi_2\mathscr{C}(D^n) \arrow[l] & \cdots \arrow[l] \\
0 & \pi_1\mathscr{C}(D^n) \arrow[l] & \pi_1\mathscr{C}(D^n) \arrow[l] & \pi_1\mathscr{C}(D^n) \arrow[l] & \cdots \arrow[l] \\
0 & \pi_0\mathscr{C}(D^n) \arrow[l] & \pi_0\mathscr{C}(D^n) \arrow[l] & \pi_0\mathscr{C}(D^n) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

In which case the differential has a simpler expression. There is an automorphism of $\mathscr{C}(D^n)$ given by "flipping along the $1/2 \in I$ line", Kupers gives an explicit map, which we will denote $\bar{f}$ for $f \in \mathscr{C}(D^n)$. Now given an element of $\pi_q\mathscr{C}(D^n)$ it is an equivalence class of elements in $\mathscr{C}(D^{n+q})$. This follows from the weak equivalence with the simplicial group $\tilde{\mathscr{C}}(D^n)$ because there an element of the q-th homotopy group is a collection of q simplicies modulo some relations coming from the $q+1$-simplicies, this can be made precise by the Moore complex. Therefore we can choose some representative, but because we are in the stable range it is therefore equally represented by an element in $\mathscr{C}(D^n)$. The involution on $\mathscr{C}(D^{n+q})$ therefore gives an involution on $\pi_q\mathscr{C}(D^n)$, but so does the involution on $\mathscr{C}(D^n)$, the relationship is characterised

Lemma ([Kup], Lem 25.3.8)**.** *Denote $\sigma : \mathscr{C}(D^n) \to \mathscr{C}(D^n \times I)$ the stabilisation map of Hatcher-Igusa and σ_q the induced map on the q-th homotopy groups. Denote $\overline{(-)}$ the involution on $\mathscr{C}(D^n)$ and $\overline{[f]}$ the induced involution on $\pi_q\mathscr{C}(D^{n+q-1})$, where $f \in \mathscr{C}(D^{n+q-1})$. Then we have that*

$$\overline{[\sigma f]} = -\sigma[\bar{f}]$$

The situation is therefore summarised by the commuting diagram, where if we assume we are in the stable range because we are modding out by the appropriate relation the vertical σ's are actually group isomorphisms:

$$\begin{tikzcd}
\pi_q\mathscr{C}(D^n) \arrow[r, "{[S^q,\overline{(-)}]}"] \arrow[d, "\sim"'] & \pi_q\mathscr{C}(D^n) \arrow[d, "\sim"] \\
\mathscr{C}(D^{n+q})/\sim \arrow[r, "\overline{(-)}"] & \mathscr{C}(D^{n+q})/\sim \\
\mathscr{C}(D^{n+q-1})/\sim \arrow[r, "\overline{(-)}"] \arrow[u, "\sigma"] & \mathscr{C}(D^{n+q-1})/\sim \arrow[u, "-\sigma"'] \\
\mathscr{C}(D^{n+q-2})/\sim \arrow[r, "\overline{(-)}"] \arrow[u, "\sigma"] & \mathscr{C}(D^{n+q-2})/\sim \arrow[u, "-\sigma"'] \\
\vdots \arrow[u, "\sigma"] & \vdots \arrow[u, "-\sigma"'] \\
\mathscr{C}(D^n)/\sim \arrow[r, "\overline{(-)}"] \arrow[u, "\sigma"] & \mathscr{C}(D^n)/\sim \arrow[u, "-\sigma"']
\end{tikzcd}$$

Now we already noted that the differential was the induced map on homotopy groups coming from the restriction to the top edge map $\delta : \mathscr{C}(D^n \times I^p) \to \mathscr{C}(D^n \times I^{p-1})$. Hatcher claims that

Lemma.

$$\delta\sigma : \mathscr{C}(D^n \times I^{p-1}) \to \mathscr{C}(D^n \times I^{p-1})$$

is given by $f \mapsto f + \bar{f}$ (addition is the group operation in the concordance group).

Looking at the induced map on π_q we get that

$$d^1 \circ \sigma_q = (\delta\sigma)_q = [S^q, \delta\sigma] = [S^q, (-) + \overline{(-)}] = [S^q, (-)] + [S^q, \overline{(-)}] = \mathrm{id}_{\pi_q\mathscr{C}(D^n\times I^{p-1})} + [S^q, \overline{(-)}]$$

But assuming we are in the stable range then every element of π_q is in the image of σ and so this constitutes a computation of the differential in the spectral sequence. Ok Putting this all together, if we wish to reduce the spectral sequence such that the rows are given by $\pi_q\mathscr{C}(D^n)$ in a stable range, then we do the following. First the differential of the new E^1 page we will call ϵ, by definition $\epsilon = (\sigma_*^{-1})^{p-1} \circ d^1 \circ \sigma_*^p$, that is use the sequence of isomorphisms σ_* to climb up to the group in the spectral sequence, apply the differential and then climb down again

$$\begin{tikzcd}
\pi_q\mathscr{C}(D^n) \arrow[r, "\sigma_*"] & \pi_q\mathscr{C}(D^n \times I) \arrow[r] & \cdots \arrow[r] & \pi_q\mathscr{C}(D^n \times I^{p-1}) \arrow[r, bend left, "\sigma_*"] & \pi_q\mathscr{C}(D^n \times I^p) \arrow[l, bend left, "d^1"]
\end{tikzcd}$$

Using the two lemmas we can algebraically rearrange this now

$$
\begin{aligned}
\epsilon &= (\sigma_*^{-1})^{p-1} \circ d^1 \circ \sigma_*^p \\
&= (\sigma_*^{-1})^{p-1} \circ (d^1 \circ \sigma)\sigma_*^{p-1} \\
&= (\sigma_*^{-1})^{p-1} \circ (\mathrm{id}_{\pi_q \mathscr{C}(D^n \times I^{p-1})} + [S^q, \overline{(-)}]) \circ \sigma_*^{p-1} \\
&= (\sigma_*^{-1})^{p-1} \circ \mathrm{id}_{\pi_q \mathscr{C}(D^n \times I^{p-1})} \circ \sigma_*^{p-1} + (\sigma_*^{-1})^{p-1} \circ [S^q, \overline{(-)}] \circ \sigma_*^{p-1} \\
&= \mathrm{id}_{\pi_q \mathscr{C}(D^n)} + (\sigma_*^{-1})^{p-1} \circ [S^q, \overline{(-)}] \circ \sigma_*^{p-1} \\
&= \mathrm{id}_{\pi_q \mathscr{C}(D^n)} + (\sigma_*^{-1})^{p-1} \circ ([S^q, \overline{(-)}] \circ \sigma_*)\sigma_*^{p-2} \\
&= \mathrm{id}_{\pi_q \mathscr{C}(D^n)} + (\sigma_*^{-1})^{p-1} \circ (-\sigma_* \circ [S^q, \overline{(-)}])\sigma_*^{p-2} \\
&= \mathrm{id}_{\pi_q \mathscr{C}(D^n)} + (-1)(\sigma_*^{-1})^{p-2} \circ [S^q, \overline{(-)}] \circ \sigma_*^{p-2} \\
&\vdots \\
&= \mathrm{id}_{\pi_q \mathscr{C}(D^n)} + (-1)^{p-1} \circ [S^q, \overline{(-)}_{\mathscr{C}(D^n)}]
\end{aligned}
$$

We have already seen however that $[S^q, \overline{(-)}_{\mathscr{C}(D^n)}] = (-1)^q \sigma_*^q \overline{(-)}_{\mathscr{C}(D^n)} \circ (\sigma_*^{-1})^q$, so if we represent an element $\gamma \in \pi_q \mathscr{C}(D^n)$ by an element $f \in \mathscr{C}(D^n)$, that is $\gamma = \sigma_*^q f$ the the differential is given by

$$
\begin{aligned}
\epsilon(\gamma) &= \mathrm{id}_{\pi_q \mathscr{C}(D^n)}(\gamma) + (-1)^{p-1} \circ [S^q, \overline{(-)}_{\mathscr{C}(D^n)}](\gamma) \\
\epsilon(\gamma) &= \gamma + (-1)^{p-1} \circ (-1)^q \sigma_*^q \overline{(-)}_{\mathscr{C}(D^n)} \circ (\sigma_*^{-1})^q(\gamma) \\
\epsilon(\gamma) &= \gamma + (-1)^{q+p-1} \circ \sigma_*^q \overline{(-)}_{\mathscr{C}(D^n)} \circ (\sigma_*^{-1})^q(\sigma_*^q f) \\
\epsilon(\gamma) &= \gamma + (-1)^{q+p-1} \circ \sigma_*^q \overline{f}_{\mathscr{C}(D^n)}
\end{aligned}
$$

so in this notation then we would be justified in writing that

$$\epsilon[f] = [f] + (-1)^{p+q-1}[\bar{f}]$$

The stick now is that computing the equivalence class of the involved element in concordance can be highly non-trivial.

Finally we have that this sequence converges to some innteresting groups

Theorem.

$$E^1_{pq} \implies \pi_{p+q+1} \frac{\widetilde{\mathrm{Diff}}_\partial(M)}{\mathrm{Diff}_\partial(M)}$$

Remark. There is an abuse of notation here $\gamma \neq \sigma_*^q f$ strictly I need to compose with the isomorphism at the very top of our diagram, but this iso is sort of tautological and moreover all diagrams commute and so no issue arrises.

Remark. In the lemma our diagram could continue down to $\mathscr{C}(D^0)/\sim$ but at some point we will lose the stable range and so σ will no longer be an isomorphism, in this case the diagram is still true however not very useful unless you known that the element you are looking at is in the image of the lower stabilisation map.

References

[Hat78] A. E. Hatcher. Concordance spaces, higher simple-homotopy theory, and applications. In R. Milgram, editor, *Proceedings of Symposia in Pure Mathematics*, volume 32.1, pages 3–21. American Mathematical Society, Providence, Rhode Island, 1978.

[Kup] Alexander Kupers. Lectures on diffeomorphism groups of manifolds, version February 22, 2019.